Solutions

Exam 1
Chapters 12 and 13
Name:______________________

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **No calculators are allowed on this exam.**

<u>**True or False**</u> (3 points each)

T **1.** The normal vector to a plane is orthogonal to every vector in the plane.

F **2.** $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.

T **3.** $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta$.

T **4.** $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} = c\mathbf{v}$ for some scalar $c \in \mathbf{R}$.

F **5.** $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$.

T **6.** A line is completely determined by a point on the line and a direction vector.

F **7.** $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}'(t)$.

F **8.** For a space curve C, the greater the curvature κ, the smaller the bend in the curve.

<u>**Show your work!**</u>

1. (3 points each)
For the following pairs of curves, determine which are parallel, orthogonal or neither.

(a) $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

(b) $\mathbf{u} = < a, b, c >$, $\mathbf{v} = < -b, a, 0 >$

(c) $\mathbf{a}$, $\mathbf{a} \times \mathbf{b}$

(a) $(-i+2j+5k)\cdot(3i+4j-k) = -3+8-5=0.$
Perpendicular

(b) $u \cdot v = -ab+ab+0=0$
perpendicular

(c) $a, a \times b$ are perpendicular by theorem on cross product.

2. Let $\mathbf{u} =< 2, 1, -3 >, \mathbf{v} =< 8, -1, 0 >$ and $\mathbf{w} =< 1, 1, 7 >$. Evaluate the following. (3 points each)

(a) $2\mathbf{u} + 1/4\mathbf{v}$

(b) $\mathbf{u} + \mathbf{v} - \mathbf{w}$

(a) $2u + \frac{1}{4}v = \langle 4+2, 2-\frac{1}{4}, -6\rangle = \langle 6, \frac{7}{4}, -6\rangle$

(b) $u+v-w = \langle 2+8-1, 1+(-1)-1, -3+0-7\rangle = \langle 9, -1, -10\rangle$

3. Determine if the following three vectors are coplanar. (5 points)

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \mathbf{c} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$$

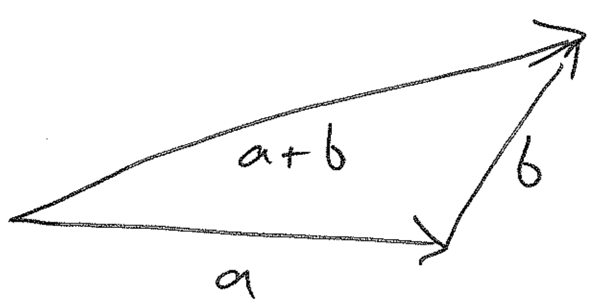
Triple scalar product

$$a\cdot(b\times c) = a\cdot\left(\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} i - \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} k\right)$$
$$= a\cdot(-2i - 2j)$$
$$= -4.$$

So non zero means not coplanar.

4. Describe in words and pictures what the vector $\mathbf{a} + \mathbf{b}$ represents given any two vectors $\mathbf{a}$ and $\mathbf{b}$. (6 points)

a+b is the vector which begins at the initial side of a and ends at the terminal side of b.



5. Let $\mathbf{P_1}$ be the plane determined by the equation $2x + 3y + 4z = 12$. (Therefore $\mathbf{n_1} =< 2, 3, 4 >$ is its normal vector.) Let $\mathbf{P_2}$ be the plane containing the two vectors $\mathbf{a} =< 1, -1, 2 >$ and $\mathbf{b} =< 0, -1, -1 >$.

(a) Find the normal vector to the plane $\mathbf{P_2}$. (5 points)

(b) Find the angle between the two planes. (5 points)

(c) Given that $(6, 0, 0)$ lies on $\mathbf{P_2}$ find the line of intersection L of $\mathbf{P_1}$ and $\mathbf{P_2}$ and write the parametric equations that define it. (5 points)

(a) $n_2 = a \times b = \langle 3, 1, -1 \rangle$

(b) $n_1 \cdot n_2 = 6 + 3 - 4 = 5$

$n_1 \cdot n_2 = |n_1||n_2| \cos\theta = \sqrt{2^2+3^2+4^2}\sqrt{3^2+1^2+1^2} \cos\theta$

$= \sqrt{319} \cos\theta$

$\theta = \cos^{-1}\left(\frac{5}{\sqrt{319}}\right)$

(c) $n_1 \times n_2 = \langle 2,3,4 \rangle \times \langle 3,1,-1 \rangle$

$= \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} i - \begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} k$

$= \langle -7, 14, -7 \rangle$ (direction vector of L)

So $x = 6 - 7t$ $y = 14t$ $z = -7t$

6. Consider the vector function given by $\mathbf{r}(t) = \langle \cos 3t, \sin 3t, 4t \rangle$.

(a) Find the length of the arc with equation $\mathbf{r}(t)$ from the point $(1, 0, 0)$ to $(1, 0, 4\pi)$. (5 points)

(b) Find the unit tangent, unit normal, and binormal vectors for $\mathbf{r}(t)$. (9 points)

(c) Find the curvature for $\mathbf{r}(t)$. (6 points)

(a) $t=0$ to $t=\pi$

$$r'(t) = \langle -3\sin 3t, 3\cos 3t, 4\rangle \qquad |r'(t)| = \sqrt{9\sin^2 3t + 9\cos^2 3t + 16} = 5$$

$$\int_0^{\pi} |r'(t)|\,dt = \int_0^{\pi} 5\,dt = 5\pi$$

(b) $$T(t) = \frac{1}{5}\langle -3\sin 3t, 3\cos 3t, 4\rangle$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\frac{1}{5}\langle -9\cos 3t, -9\sin 3t, 0\rangle}{\frac{1}{5}\sqrt{81\cos^2 3t + 81\sin^2 3t}} = \frac{\frac{1}{5}\langle -9\cos 3t, -9\sin 3t, 0\rangle}{9/5} = \langle -\cos 3t, -\sin 3t, 0\rangle$$

$$B(t) = T(t)\times N(t) = \frac{1}{5}\left(\begin{vmatrix} 3\cos 3t & 4 \\ -\sin 3t & 0 \end{vmatrix} i - \begin{vmatrix} -3\sin 3t & 4 \\ -\cos 3t & 0 \end{vmatrix} j + \begin{vmatrix} -3\sin 3t & 3\cos 3t \\ -\cos 3t & -\sin 3t \end{vmatrix} k \right)$$

$$= \frac{1}{5}\left(-4\sin 3t\, i - 4\cos 3t\, j + 3k \right)$$

(c) $$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{9/5}{5} = \frac{9}{25}$$

7. Let $\mathbf{r}(t) = t \sin t \; \mathbf{i} + t \cos t \; \mathbf{j} + t^2 \; \mathbf{k}$ be the position function for a particle moving through space. Find the velocity, acceleration, and speed of the particle in general and at time 5π . (15 points)

$$v(t) = r'(t) = (\sin t + t\cos t)i + (\cos t - t\sin t)j + 2t\,k$$

$$a(t) = v'(t) = (2\cos t - t\sin t)i + (-2\sin t - t\cos t)j + 2\,k$$

$$s(t) = \text{speed} = |v(t)| = \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + (2t)^2}$$

$$= \sqrt{\sin^2 t + t^2\cos^2 t + 2t\sin t\cos t + \cos^2 t + t^2\sin^2 t - 2t\sin t\cos t + 4t^2}$$

$$= \sqrt{1 + 5t^2}$$

$$v(5\pi) = -5\pi i - \cancel{1}j + 10\pi k$$

$$a(5\pi) = -2i + 5\pi j + 2k$$

$$s(5\pi) = \sqrt{1 + 125\pi^2}$$

Extra Credit

Answer one of the following two extra credit questions.

1. Name one person you would like to meet. (15 points)

2. Kublai Khan is beginning his assault on the city of Xiangyang. Along with his world-renowned cavalry, he also has a dozen trebuchets. Before him stands the 15 m high wall of the city stretching as far as the eye can see in the dim morning light. He sets the trebuchets down 100 m from the wall, well within range for the trebuchets, but out of range of the enemy archers on the wall. Kublai turns to the commander of his army (that's you!) and asks, "At what range of angles can we release the rocks to clear the wall?" As his commander you know you must answer and answer correctly or endure his wrath. You recall that the rocks will leave the trebuchet at an initial speed of 80 m/s, gravity acts at a rate of 9.8 m/s^2. Using your expertise as general you notice that the air is still and you see that the land is flat before and the walls rise perpendicular to the ground. What do you tell him? (Kublai expects preciseness and has a brain like a calculator. Therefore you can leave your answer in terms of inverse trig functions with nasty expressions on the inside.) (15 points)